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Note

Numerical Instability Due to the Lorentz Force in a Relativistic Fluid, Laser–Plasma Code

This note investigates a numerical instability due to including $\mathbf{v} \times \mathbf{B}$ in a one-dimensional, relativistic fluid (with fluid velocity $\mathbf{v}$), Eulerian finite difference scheme based on the method of characteristics [1] and the advective differencing method due to Langdon and Dawson [2] for Maxwell's curl equations. The effect of $\mathbf{v} \times \mathbf{B}$ is to preclude a solution solely via characteristics for the self-consistent EM fields in a cold, collisionless plasma. Longitudinal charge motion introduces a (charge) conservation law and thus the system of equations is nonhyperbolic. We derive the equations of first variation for the most natural generalization for treating $\mathbf{v} \times \mathbf{B}$, known as "Lelevier's remedy" [3], upwind (downwind) or full donor cell differencing [4], and show these imply a conditional stability which depends on the local electron fluid velocity and plasma density, but not on the dimensionless time step $\omega \Delta t$. This instability, which is present for any donor cell differencing scheme, is distinctly different from the numerical Cherenkov instability which Godfrey [5] and Godfrey and Langdon [6] studied for the particle-code implementation of the Langdon–Dawson scheme, since error growth here is associated with electron orbit turning points rather than fast electron-EM wave resonance.

Figure 1 shows the geometry we consider. A linearly polarized pulse is normally incident on a plasma filling the half-space $z > 0$ with an arbitrary initial charge density. Now if one simply ignores the $\mathbf{v} \times \mathbf{B}$ force on plasma electrons (we consider ions to be immobile over the time periods of interest) the Maxwell's equations and the relativistic Lorentz force equation

$$(d/dt)\, m\mathbf{v}/(1 - v^2/c^2)^{1/2} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c) \tag{1}$$

form a quasilinear hyperbolic system, readily integrable via the method of characteristics [7, 8]. In this case only transverse motion is allowed, but the plasma electrons radiate according to Maxwell's equations and have relativistic mass.

On replacing $\mathbf{v} \times \mathbf{B}$, we shall see that each equation may be written in normal, or characteristic form except one, the conservation law for charge. We shall difference this by replacing the space derivative by a forward or backward space difference, according to the sign of the local v_z. The resulting explicit difference scheme is only conditionally stable. We shall derive stability criteria for the case of linearly polarized incident light, frozen ions, and full donor cell differencing; however, the structure of the system of equations is only slightly modified and our conclusions can be extended without essential modification for the case of arbitrary elliptic polarization, mobile ions, and any fractional donor cell differencing method.

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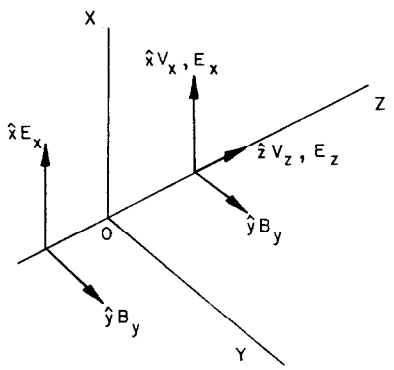

FIG. 1. A planewave, linearly polarized along x, is normally incident on a plasma slab occupying the half-space $z > 0$. Ions are assumed to be immobile and electrons move in x and z.

Although retaining the $\mathbf{v} \times \mathbf{B}$ term rules out a numerical integration via characteristics alone, the method characteristics is ideally suitable for the case without $\mathbf{v} \times \mathbf{B}$. Thus a number of interesting nonlinear optical effects may be studied, at least qualitatively, in particular nonlinear pulse reflection. IRRAD [8], a Pl/1 version of the algorithm, is currently being used to study intensity-dependent penetration and partial trapping of reflected pulses. A brief discussion of stability is included in the Appendix.

For the geometry of Fig. 1, Maxwell's equations and (1) can be written

$$\begin{aligned}
(\partial/\partial t \pm c\,\partial/\partial z)\,E_\pm &= (\omega/\sqrt{2})\,n' v_x', \\
(\partial/\partial t + cv_z'\,\partial/\partial z)\,E_z' &= \omega n_0' v_z', \\
(\partial/\partial t + cv_z'\,\partial/\partial z)\,u_x' &= -(\omega/\sqrt{2})[E_+(1 - v_z') + E_-(1 + v_z')], \\
(\partial/\partial t + cv_z'\,\partial/\partial z)\,u_z' &= -\omega[E_z' + (E_+ - E_-)\,v_x'/\sqrt{2}], \\
(\partial/\partial t)\,n' + (\partial/\partial z)\,n' c v_z' &= 0,
\end{aligned} \tag{2}$$

with the following definitions of the dimensionless variables (in *cgs-esu*):

$$\begin{aligned}
E_\pm &= (-e/mc\omega)(E_x \pm B_y)/\sqrt{2}, \\
E_z' &= (-e/mc\omega)\,E_z, \\
\mathbf{u}' &= \mathbf{u}/c = \mathbf{v}/c(1 - v^2/c^2)^{1/2} = \mathbf{v}'(1 - v'^2)^{1/2}, \\
n' &= 4\pi e^2 n/m\omega^2, \\
n_0' &= 4\pi e^2 n_0/m\omega^2,
\end{aligned} \tag{3}$$

where $-e$ is the electron charge, n the electron number density, $n_0 = n_0(z)$ the ion number density, m the electron mass, and ω the incident EM wave (angular) frequency.

The first five of Eqs. (2) are in normal, or characteristic form,

$$(\partial/\partial t + c\beta\ \partial/\partial z)f = \omega g$$

and may be differenced by replacing directional derivatives with differences along segments of the "characteristics." We use a grid with $\Delta z = c\Delta t$, place all variables at grid points $(n\Delta t, j\Delta z)$, and, following Courant *et al.* [1], replace each equation in normal form by

$$f_j^{n+1} = f_Q^{\,n} + (\omega\ \Delta t)\, g_j^{\,n}. \tag{4}$$

Here Q denotes the intersection point where the segment with $dz/dt = c\beta$ drawn backwards from $(j\Delta z,(n+1)\,\Delta t)$ crosses $t = n\,\Delta t$, and $f_Q{}^n$ is obtained by linear interpolation:

$$\begin{aligned} f_Q{}^n &= f_j^{\,n} - \beta_j{}^n(f_j^{\,n} - f_{j-1}^{\,n}), \qquad \beta_j{}^n \geqslant 0, \\ f_Q{}^n &= f_j^{\,n} - \beta_j{}^n(f_{j+1}^{\,n} \quad f_j^{\,n}), \qquad \beta_j{}^n < 0. \end{aligned} \tag{5}$$

In the sixth of Eqs. (2) we use a backward (forward) space difference to replace $\partial(n'v_z')/\partial z$ if $v_j{}^n \geqslant 0$ (< 0). Thus

$$n_j'^{n+1} = n_j'^{n} - J_\pm{}^n((v_z)_j^{\,n} \geqslant 0, 0), \tag{6}$$

where

$$\begin{aligned} J_+{}^n &= (n'v_z')_j^{\,n} - (n'v_z')_{j-1}^{\,n}, \\ J_-{}^n &= (n'v_z')_{j+1}^{\,n} - (n'v_z')_j^{\,n}. \end{aligned} \tag{7}$$

This scheme avoids the use of differences over the double space interval $2\Delta x$. The latter would yield symmetric difference equations, but leads to unconditional instability in closely related Eulerian schemes for one-dimensional fluid dynamics [3]. Our scheme, like "Lelevier's remedy" [3], includes flow only from the neighboring grid point on the left (right) if $v_{zj}^n \geqslant 0$ (< 0).

In any actual computation the difference equations would include special-case tests to follow charge depletion regions, but the following discussion does not rely on their nature.

Let $\mathbf{F}$ denote the exact solution of the finite difference equations (4)–(7),

$$\mathbf{F}_j{}^n = (E_+, E_-, E_z', u_x', u_z', n')_j^{\,n}, \tag{8a}$$

and $\tilde{F}_j{}^n$ the solution of the differential equations evaluated at $z = j\,\Delta t$, $t = n\,\Delta t$,

$$\tilde{F}_j{}^n = (E_+, \ldots, n')|_{z=j\Delta t, t=n\Delta t}. \tag{8b}$$

The error vector $\epsilon = \mathbf{F} - \tilde{F}$ evolves according to

$$\begin{aligned}\epsilon_j^{n+1} &= \mathbf{F}_j^{n+1} - \tilde{F}_j^{n+1} \\ &= \sum_{k=j-1}^{j+1} \alpha_{jk}^n \cdot \epsilon_k{}^n + O(\epsilon^2) + O(\omega^2 \Delta t^2),\end{aligned} \tag{9}$$

where α_{jk}^n is the Jacobian of the evolution operator evaluated at $t = n\,\Delta t$:

$$\partial(\mathbf{F}_j^{n+1})_r / \partial(\mathbf{F}_k{}^n)_s = (\alpha_{jk}^n)_{rs} + O(\Delta t). \tag{10}$$

Equation (9) may be derived from the "consistency condition"

$$\lim_{\Delta t \to 0} \left| \frac{\mathbf{M}(\tilde{F}_j{}^n) - \tilde{F}_j{}^n}{\Delta t} - (\partial \tilde{F}/\partial t)_j^n \right| = 0, \tag{11}$$

where $\mathbf{M}$, the evolution operator, is a shorthand for (4)–(7):

$$\mathbf{F}_j^{n+1} = \mathbf{M}(\mathbf{F}_{j-1}^n, \mathbf{F}_j{}^n, \mathbf{F}_{j+1}^n) \equiv \mathbf{M}(\mathbf{F}_j{}^n). \tag{12}$$

One may verify (11) straightforwardly by inserting a Taylor series expansion about $j\Delta z$ for each appearance of $\tilde{F}_{j\pm1}$ in the operator $\mathbf{M}$ in (11), and then using (2) to cancel the components of the quotient and derivative terms in (11).

From (11), one finds

$$\begin{aligned}\epsilon_j^{n+1} &= \mathbf{M}(\tilde{F}_j{}^n + \epsilon_j{}^n) - \tilde{F}_j^{n+1} \\ &= \mathbf{M}(\tilde{F}_j{}^n) - \tilde{F}_j^{n+1} + \sum_{k=j-1}^{j+1} \alpha_{jk}^n \cdot \epsilon_k{}^n + O(\epsilon^2) \\ &= \sum_{k=j-1}^{j+1} \alpha_{jk}^n \cdot \epsilon_k{}^n + O(\epsilon^2) + O((\omega\,\Delta t)^2)\end{aligned}$$

as claimed.

Performing the differentiation in (10), we find

$$\alpha_{jk}^n = \begin{pmatrix} D_{jk}^n & 0 \\ 0 & L_{jk}^n \end{pmatrix},$$

where D and L denote the 3×3 matrices in the Appendix. The D's are diagonal, with eigenvalues less than unity in magnitude. Hence only the L's may cause error growth. Since the L's are in general nondiagonalizable, one cannot directly apply the von Neumann criterion, even locally.

To determine the stability criteria we define the norm of a matrix L as

$$\| L \| = \max_{|\hat{r}|=1} \hat{r} \cdot L \cdot \hat{r} \tag{13}$$

and claim that instability is equivalent to $\| L_{jk}^n \| > 1$. To see that the scheme is conditionally stable it suffices to consider L_{jj}^n. We find

$$\| L_{jj}^n \| = \{1 \mp v_z' + \tfrac{1}{2}n'(1 - \mathbf{v}'^2)^{1/2}(1 - 2v_z'^2 + v_z'^4 + v_x'^2 v_z'^2)^{1/2}\}_j^n \tag{14}$$

which can exceed unity. Figure 2 shows the regions of v-space corresponding to instability for several choices of the dimensionless electron density n'.

Thus, local error growth is possible, but note that the *powers* of L (which one should expect to describe *accumulated* error growth) are

$$L^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & 0 \\ an\lambda^{n-1} & bn\lambda^{n-1} & \lambda^n \end{pmatrix}. \tag{15}$$

Hence $L^n \to 0$ as $n \to \infty$. This does not imply stability, but only that the effect of errors is screened after a number of steps. Since (14) shows errors may grow initially, we expect the linearized analysis will break down and the exponential decay implied by (15) will not in fact dominate.

This instability will also be present for any "fractional donor cell" differencing scheme which replaces (6)–(7) by an equation of the form

$$n_j'^{n+1} = n_j'^n + \sum_{rs} \sum d_{rs} n_r'^n (v_z')_s^n , \tag{16}$$

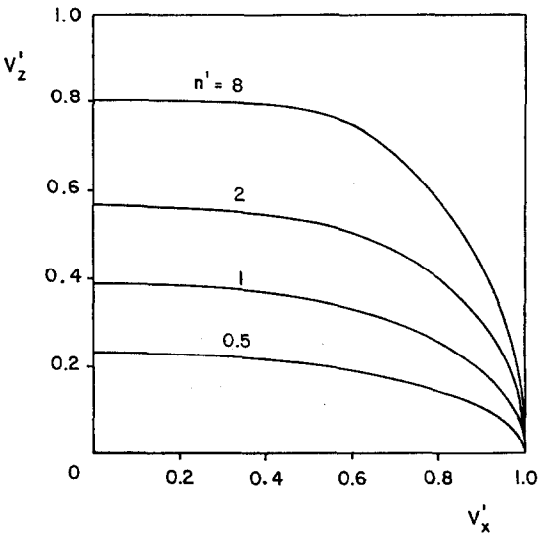

FIG. 2. Regions of instability in the $v_x' - v_z'$ plane (from Eq. (14)). The curves are labeled by the dimensionless electron density $n' = 4\pi n_e e^2/m\omega^2$, where n_e is the instantaneous electron number density. Regions of instability lie *inside* each contour. These curves show that for any nonzero electron density error amplification will occur in the vicinity of the turning points of the electron orbits ($v_z = 0$).

where the quantities d_{rs} depend on local averages of v'_z values or on the signs of local v'_z values (see [4] for example). If (16) replaces (6)–(7) the only change in the relevant error growth matrices L_{jk} will be in the third row, and will involve

$$\partial n'^{n+1}_j/\partial u'^n_{xk}\,, \qquad \partial n'^{n+1}_j/\partial u'^n_{zk}\,, \qquad \partial n'^{n+1}_j/\partial n'^n_k.$$

Note that the first two of these are simply scaled by the coefficient d_j, (we only need consider L_{jk}) from their previous values, suggesting that (16) amounts to a relabeling of the n' contours in Fig. 2. The (3, 3) element is not simply scaled because of the leading term in (16). Thus the new norm cannot be written by inspection of (14). Near $v'^n_{zj} \simeq 0$, however, where Figure 2 shows instability for any n', the (3, 3) element is effectively unity for either differencing scheme, and the "scaling rule" is valid. Hence the instability regions may move to different depths in the plasma, but will not be removed, by (16).

For the case of arbitrary elliptic polarization, the system of equations is modified to include $F_\pm \equiv -(e/mc\omega)(E_y \mp B_x)/2^{1/2}$ and u'_y as follows:

$$(\partial/\partial t \pm c\,\partial/\partial z)\,F_\pm = (\omega/\sqrt{2})\,n'v'_y\,,$$
$$(\partial/\partial t + cv'_z\,\partial/\partial z)\,u_y = -(\omega/\sqrt{2})[F_+(1 - v'_z) + F_-(1 + v'_z)].$$

Letting the new solution vector be

$$F = (E_+\,, E_-\,, F_+\,, F_-\,, E'_z\,, u'_x\,, u'_y\,, u'_z\,, n'),$$

and differencing each equation in normal form along characteristics as before, one will find the 4×4 submatrix in the new 9×9 error growth matrix α_{jj} which now governs stability is

$$L^n_{jj} = \begin{pmatrix} \lambda_\pm & 0 & 0 & 0 \\ 0 & \lambda_\pm & 0 & 0 \\ 0 & 0 & \lambda_\pm & 0 \\ \pm p & \pm q & \mp r & \lambda_\pm \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} p &= [n'v'_x v'_z(1 - v^2)^{1/2}]^n_j\,, \\ q &= [n'v'_y v'_z(1 - v^2)^{1/2}]^n_j\,, \\ r &= [n'(1 - v'^2)(1 - v'^2_z)]^n_j\,. \end{aligned} \tag{18}$$

The norm of L^n_{jj} may be written

$$\| L^n_{jj} \| = \max_{|s|=1} \{\lambda_\pm + (ps_1 + qs_2 - rs_3)\,s_4\}.$$

This exceeds unity, since it reduces to the previous form if we set $s_2 = 0$.

Finally we note that permitting ions to move adds to the solution vector three ion velocity components plus the ion density. The normal form equations are differenced

as before, and the conservation law for ions analogously to (6)–(7). The error growth matrices now contain an additional 4×4 matrix on the main diagonal identical in structure to (17)–(18), but with ion density and velocity everywhere replacing the electron density and velocity.

Appendix A: Stability of the Difference System with $\mathbf{v} \times \mathbf{B} = 0$

If one arbitrarily sets $\mathbf{v} \times \mathbf{B} = 0$ in the basic system (2), and replaces the reduced system,

$$\begin{aligned} (\partial/\partial t \pm c\,\partial/\partial z)\, E_{\pm} &= (\omega/\sqrt{2})\, n' v'_x\,, \\ (\partial/\partial t)\, u'_x &= -(\omega/\sqrt{2})(E_+ + E_-), \end{aligned} \tag{A-1}$$

by the difference system as in (4),

$$\begin{aligned} E^{n+1}_{\pm j} &= E^n_{\pm j \mp 1} + (\omega\, \Delta t/\sqrt{2})(n' v'_x)^n_j\,, \\ u'^{n+1}_{xj} &= u'^n_{xj} - (\omega\, \Delta t/\sqrt{2})(E_+ + E_-)^n_j\,, \end{aligned} \tag{A-2}$$

then the error growth matrices, defined analogously to α in (10), are

$$\alpha^n_{j,j-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \alpha^n_{jj} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha^n_{j,j+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The norms of these matrices do not exceed unity; the scheme (A-2) is stable in the sense of (13). The code IRRAD [7, 8] replaces the source current terms on the right-hand sides of (A-2) by their averages at the endpoints of the characteristic segments. This does not alter the α's above, as can be seen by direct (implicit) differentiation of the difference equations.

Appendix B

The submatrices in (16) are

$$D^n_{j,j-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v'^n_{zj} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L^n_{j,j-1} = \begin{pmatrix} v'^n_{zj} & 0 & 0 \\ 0 & v'^n_{zj} & 0 \\ -a^n_{j-1} & b^n_{j-1} & v'^n_{zj-1} \end{pmatrix}, 0,$$

$$D_{jj}^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_\pm \end{pmatrix}, \qquad L_{jj}^n = \begin{pmatrix} \lambda_\pm & 0 & 0 \\ 0 & \lambda_\pm & 0 \\ \pm a_j^n & \mp b_j^n & \lambda_\pm \end{pmatrix},$$

$$D_{j,j+1}^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -v_{zj}'^n \end{pmatrix},$$

$$L_{j,j+1}^n = 0, \begin{pmatrix} -v_{zj}'^n & 0 & 0 \\ 0 & -v_{zj}'^n & 0 \\ a_{j+1}^n & -b_{j+1}^n & v_{zj+1}'^n \end{pmatrix}.$$

Here the first and second entries apply when $v_{zj}'^n \geqslant 0$ and $v_{zj}'^n < 0$, respectively. We have defined

$$a_j^n = [n'(1 - \mathbf{v}'^2)^{1/2} v_x' v_z']_j^n ,$$

$$b_j^n = [n'(1 - \mathbf{v}'^2)^{1/2}(1 - v_z'^2)]_j^n ,$$

$$\lambda_\pm = 1 \mp v_{zj}'^n .$$

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RICHARD C. SHOCKLEY†
Naval Ocean Systems Center
San Diego, California 92152

† National Research Council Resident Research Associate. Present Address: Naval Research Laboratory, Washington, D. C. 20375.